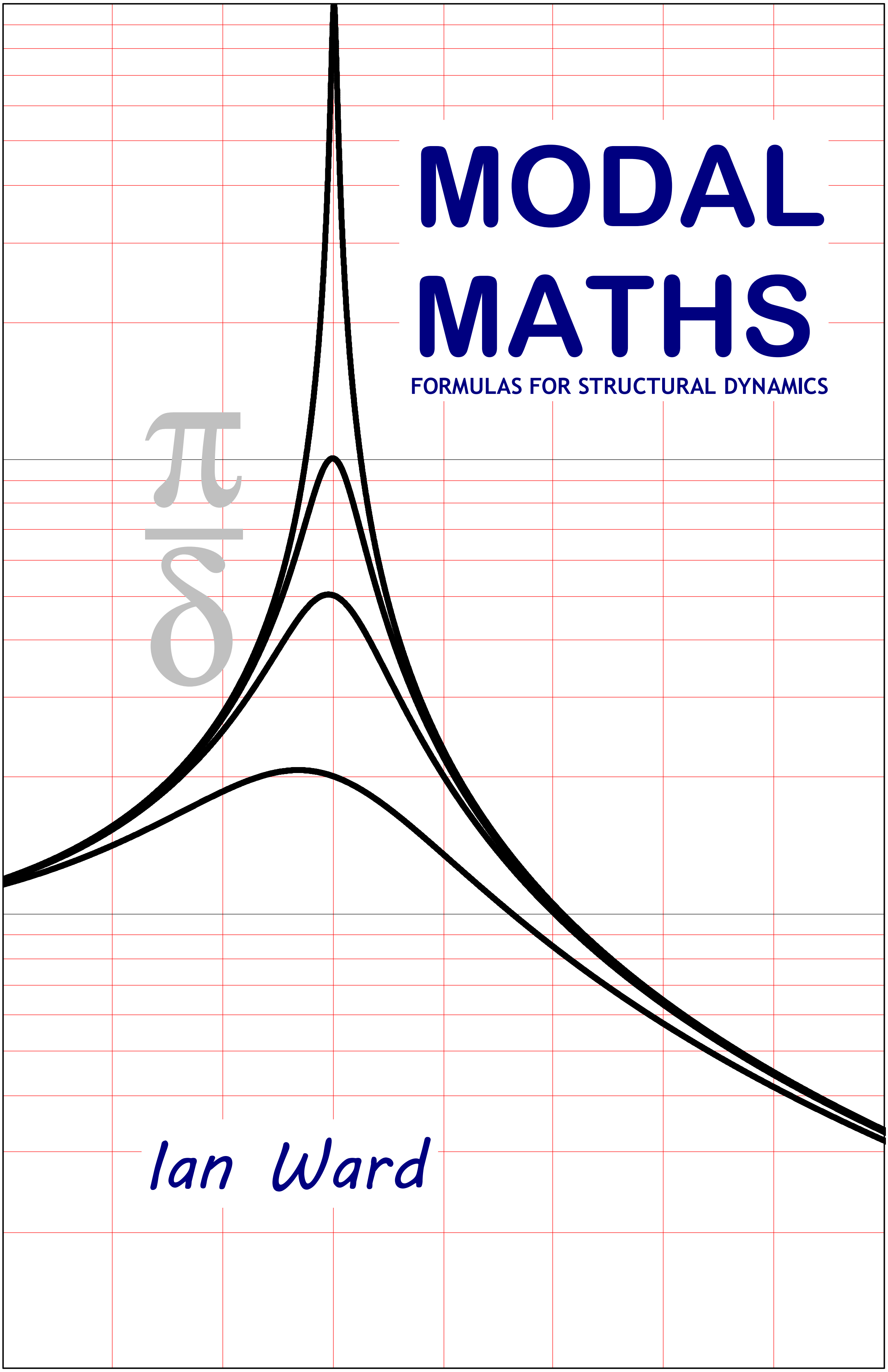
MODAL MATHS

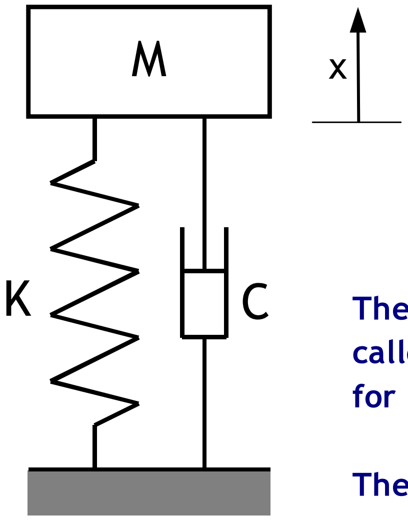
FORMULAS FOR STRUCTURAL DYNAMICS

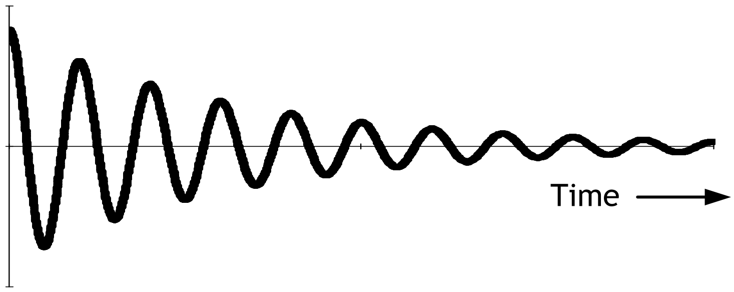


Ian Ward

Natural frequency

The single degree of freedom, or SDOF, system is a useful concept in structural dynamics. It consists of a mass M connected to ground via a spring with stiffness K and a damper with damping coefficient C. If the mass is given an initial displacement (x) and then released, the displacement oscillates about zero and gradually decreases with time.





The frequency of oscillation is called the natural frequency and, for low levels of damping, is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$$

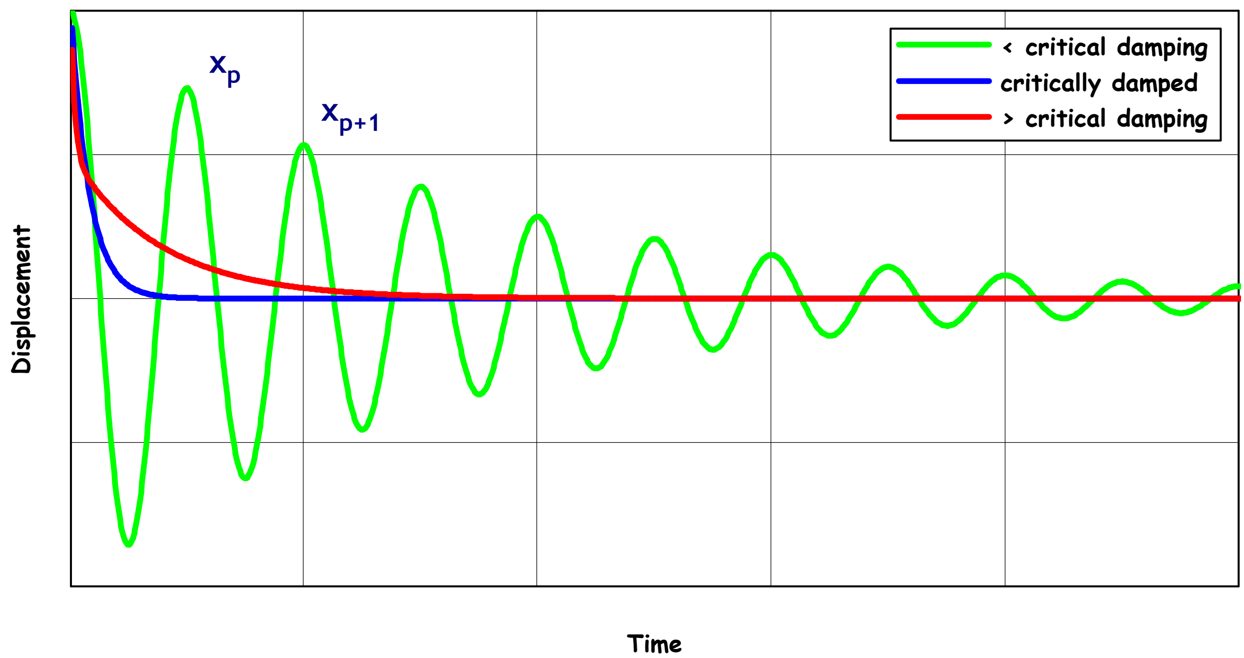
The period of oscillation is

$$T = \frac{1}{f_n}$$

The rate of decay of the response is governed by damping. For structural applications this is expressed as a 'damping ratio' or as a 'logarithmic decrement'.

Damping Ratio

If the damping were to be gradually increased until the displacement dropped to zero without any oscillation then the SDOF would be said to be 'critically damped' with a damping coefficient of C_{crit} (indicated by the blue curve in the graph below).



Any further increase in the damping would give rise to a slower decay to zero (the red curve in the graph). In practice, structural damping is only a fraction of the critical value and the decay of vibration is more closely represented by the green curve in the graph. The amount of damping can be defined in terms of a critical damping ratio:

damping ratio $$\xi = \frac{C}{C_{crit}}$$

The relationship between the damping ratio and the damping coefficient is

$$C = 2\xi M\omega = 2\xi\sqrt{MK}$$

with the circular frequency ω given by $\omega = 2\pi f$

<u>Logarithmic Decrement</u>

An alternative way of describing the structural damping is to consider the height of successive peaks in the vibration decay (denoted as x_p and x_{p+1} in the graph on the previous page). The natural logarithm of this ratio is the logarithmic decrement δ (or 'log. dec.'):

logarithmic decrement $$\delta = \ln\left(\frac{x_p}{x_{p+1}}\right) \quad \text{i.e.} \quad \frac{x_p}{x_{p+1}} = e^{\delta}$$

Or, if the decay over a number of cycles N is considered then

logarithmic decrement $$\delta = \frac{1}{N}\ln\left(\frac{x_p}{x_{p+N}}\right) \quad \text{i.e.} \quad \frac{x_p}{x_{p+N}} = e^{N\delta}$$

The relationship between the logarithmic decrement and the damping ratio is

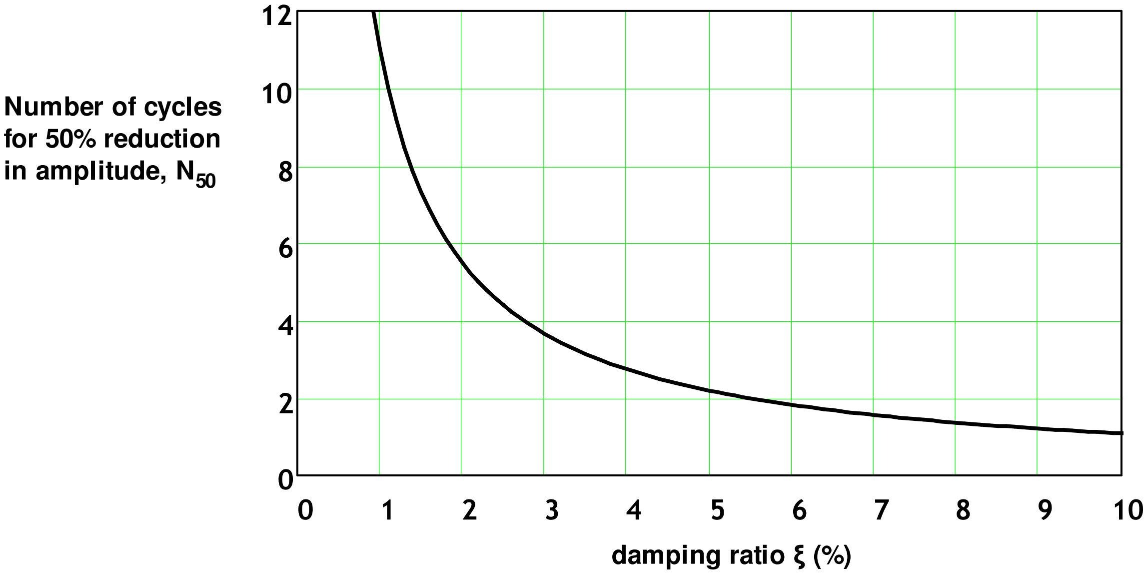
$$\delta = 2\pi\xi$$

As a rough guide, a logarithmic decrement of 0.1 means that the peak amplitude falls by approximately 10% in each successive cycle.

Another way of visualizing the vibration decay associated with a particular damping value is by showing that the number of cycles required to cause the peak amplitude to decay by, say, 50% is:

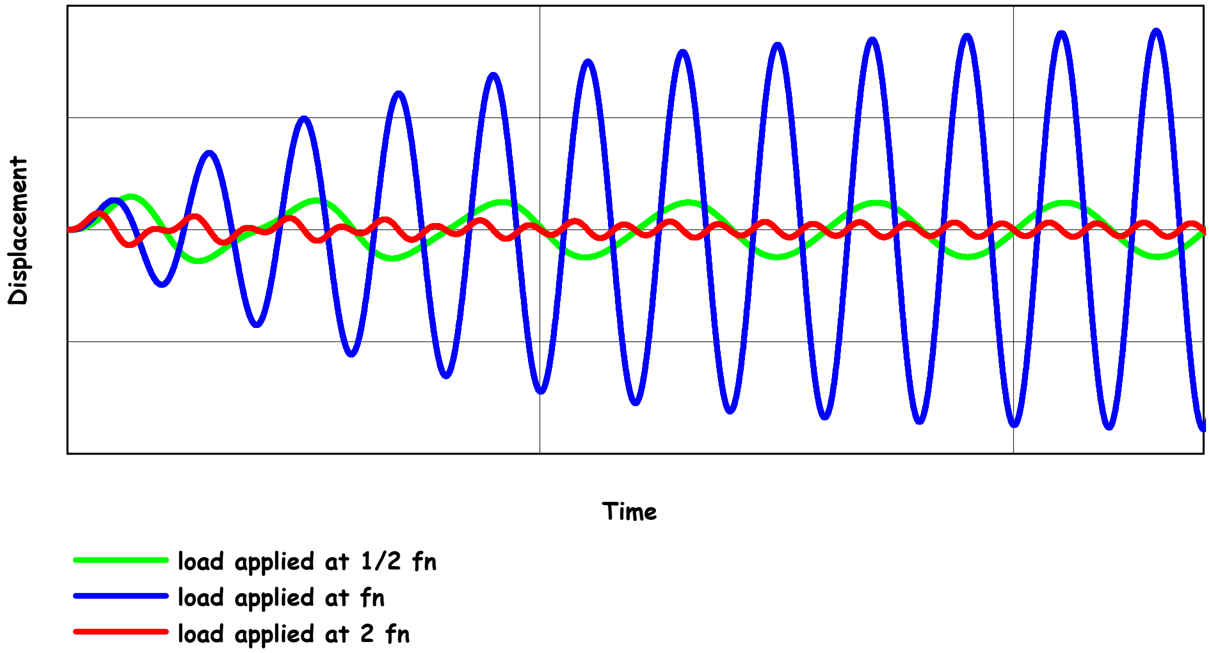
$$N_{50} = \frac{\ln\left(\frac{1}{0.5}\right)}{\delta} = \frac{\ln(2)}{\delta} = \frac{\ln(2)}{2\pi\xi}$$

(this relationship is shown in the graph overleaf)



Dynamic Magnification Factor

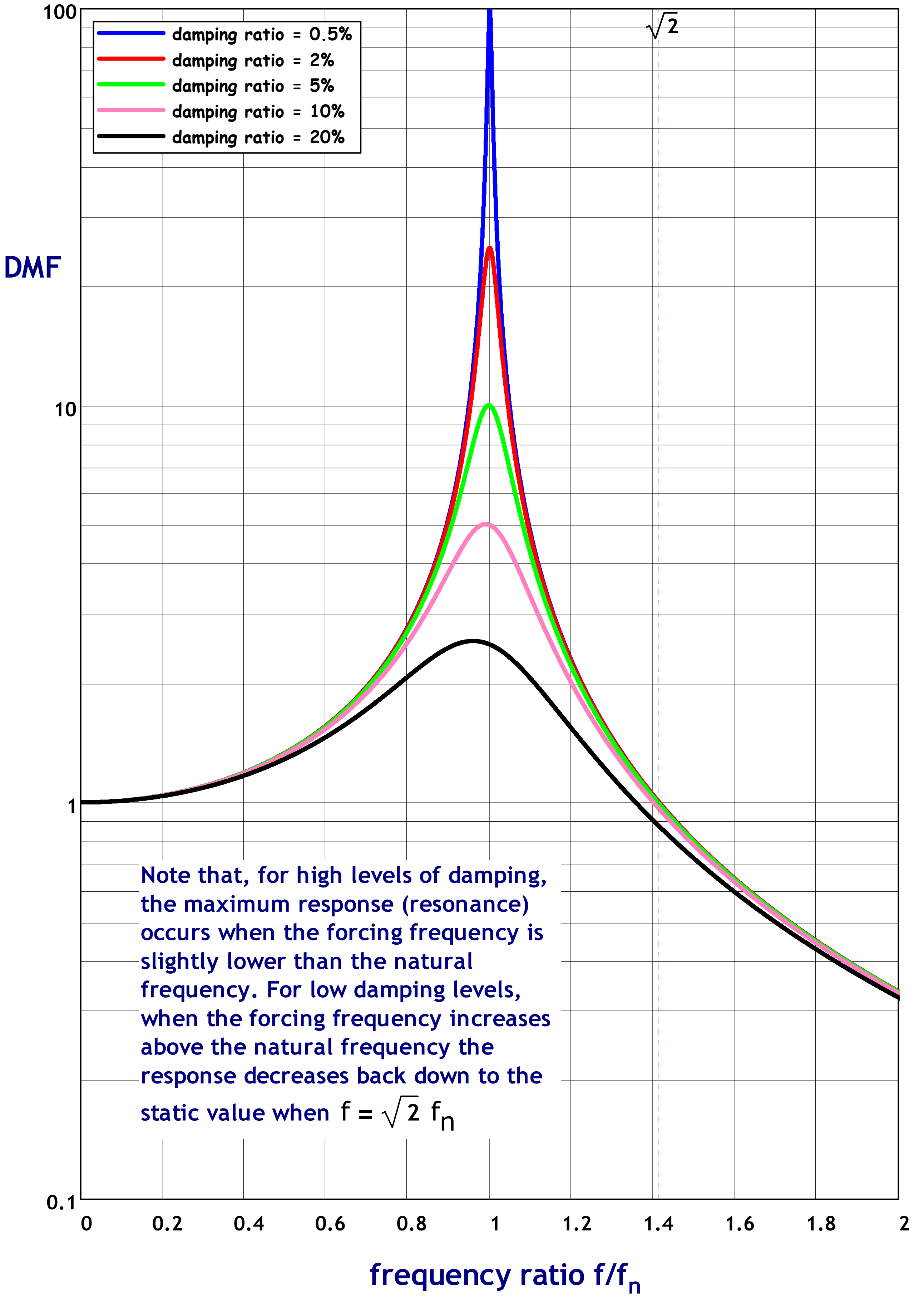
If a sinusoidal load is applied to the SDOF the response eventually reaches a 'steady-state' condition. The graph below shows the response of a SDOF with 5% damping ratio when the sinusoidal load is applied at three different frequencies (but with the same load amplitude).



The peak value of the steady-state deflection, as a proportion of the static deflection, is called the 'Dynamic Magnification Factor', given by:

$$DMF = \frac{1}{\sqrt{\left[1 - \left(\frac{f}{f_n}\right)^2\right]^2 + \left(2\xi \frac{f}{f_n}\right)^2}}$$

DMF versus frequency ratio for different levels of damping



The maximum DMF occurs when the forcing frequency is

$$f_{max} = f_n\sqrt{1 - 2\xi^2}$$

The value of the maximum DMF is

$$DMF_{max} = \frac{1}{2\xi} = \frac{\pi}{\delta}$$

From this it follows that the displacement at resonance is

$$x_{res} = \frac{1}{2\xi}\frac{F}{K} = \frac{\pi}{\delta}\frac{F}{K}$$

and the acceleration at resonance is

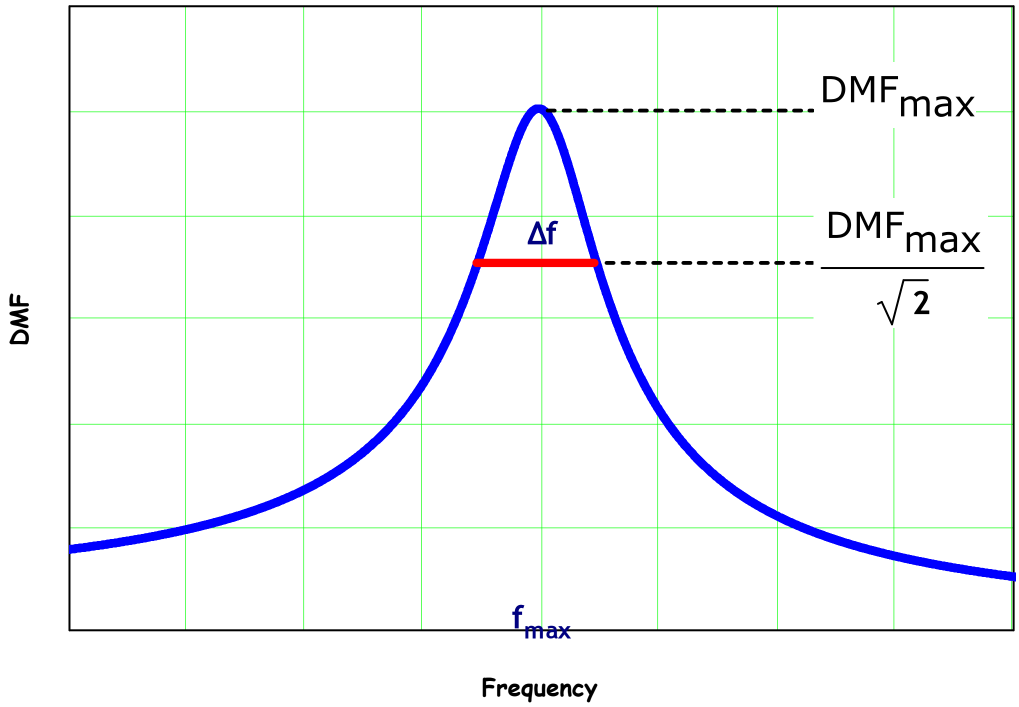
$$a_{res} = \frac{1}{2\xi}\frac{F}{M} = \frac{\pi}{\delta}\frac{F}{M}$$

The DMF curve also supplies a means for determining the damping ratio:

$$\xi = \frac{\Delta f}{2f_{max}}$$

where Δf is the width of the DMF curve at $\frac{1}{\sqrt{2}}$ times the resonant amplitude at frequency f_{max}.

This is referred to as the half-power bandwidth method for determining the damping, and is shown graphically in the figure below.

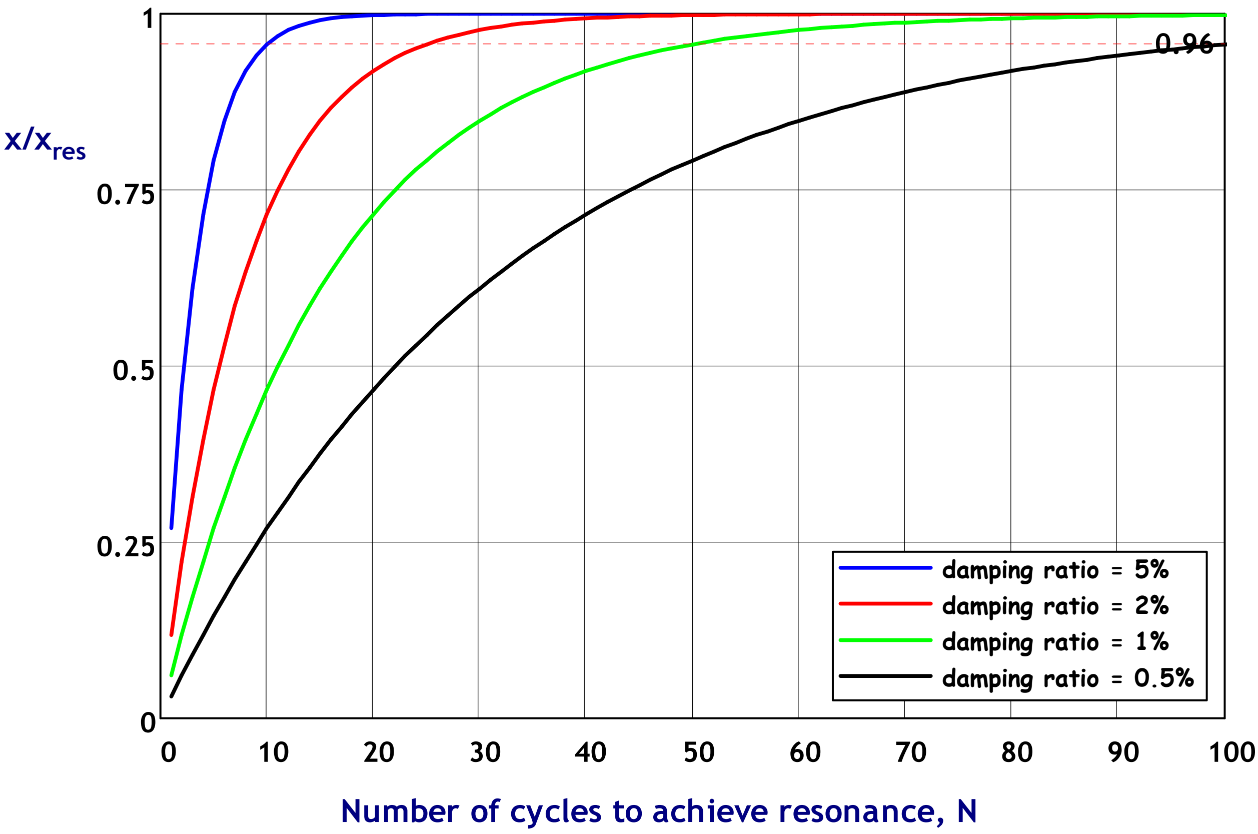


Build-up of Resonant Response

The response after a number of cycles, N, as a proportion of the final resonant response is given by:

$$\frac{x}{x_{res}} = \left(1 - e^{-2\pi\xi N}\right)$$

This expression is illustrated in the figure below for a number of different damping ratios.



An approximate relationship for the number of cycles required to reach the maximum resonant response (strictly speaking, 96% of it) is:

$$N_{peak} = \frac{1}{2\xi} = \frac{\pi}{\delta}$$

The figure shows that roughly the same number of cycles again are required for the response to build up from 96% to 100% of the maximum resonant response.

Formulas for natural frequency

Undamped natural frequency of system with stiffness K and mass M

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$$

Damped natural frequency

$$f_d = f_n\sqrt{1-\xi^2}$$

(This shows that the damped natural frequency of a structure with 5% damping will only be 0.1% lower than the undamped natural frequency. This means that, for typical engineering structures, it can be assumed that $f_d = f_n$).

Natural frequency of a SDOF system in terms of the self-weight deflection Δ caused by 1g

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

i.e. $$f_n = \frac{15.76}{\sqrt{\Delta}}$$ (with Δ in mm)

Modified version of the equation above. Approximate value for the natural frequency of a structure with distributed mass and stiffness

$$f_n = \frac{18}{\sqrt{\Delta}}$$ (with Δ in mm)

String/cable of length L with tension T and mass per unit length m

$$f_n = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Pendulum of length L

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Undamped natural frequency of system with torsional stiffness K (moment/rotation) and mass moment of inertia J.

$$f_n = \frac{1}{2\ \pi}\sqrt{\frac{K}{J}}$$

Approximate formula for road bridge of length L metres

$$f_n = \frac{100}{L}$$

Approximate formula for building of height H metres

$$f_n = \frac{46}{H}$$

Modal properties of uniform beams with various support conditions

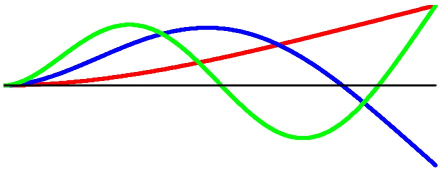
Natural frequency of mode i

$$f_i = \frac{\lambda_i^2}{2\pi L^2}\sqrt{\frac{EI}{m}}$$

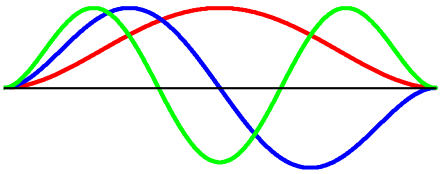
E=Young's modulus, L=length, I=area moment of inertia, m=mass per unit length. (see below for values of λ)

Mode	λ	Generalized mass (fraction of total mass)	Participation factor
Fixed - Free beam			
i = 1	1.875	0.250	1.566
i = 2	4.694	0.250	0.867
i = 3	7.855	0.250	0.509
Fixed - Fixed beam			
i = 1	4.730	0.396	1.320
i = 2	7.853	0.439	0.000
i = 3	10.996	0.437	0.550
Fixed - Pinned beam			
i = 1	3.927	0.439	1.298
i = 2	7.069	0.437	0.125
i = 3	10.210	0.437	0.506
Pinned - Pinned beam			
i = 1	π	0.500	1.273
i = 2	2π	0.500	0.000
i = 3	3π	0.500	0.424
Free - Free beam			
i = 1	4.730	0.250	0.000
i = 2	7.853	0.250	0.000
i = 3	10.996	0.250	0.000

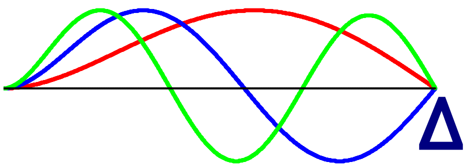
Fixed - Free beam



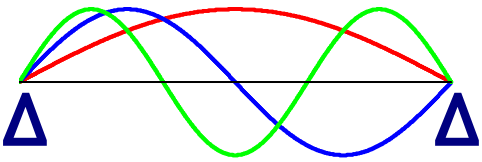
Fixed - Fixed beam



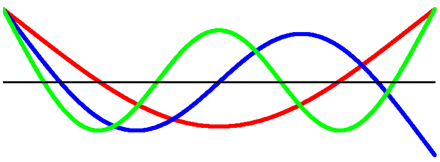
Fixed - Pinned beam



Pinned - Pinned beam



Free - Free beam

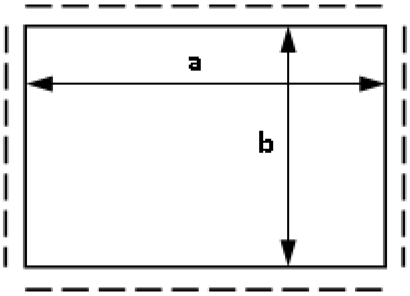


Natural frequencies of plates with various support conditions

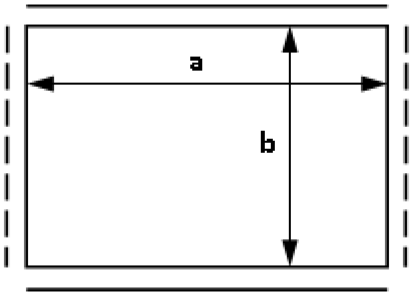
Natural frequency

$$f = \frac{\lambda}{a^2}\sqrt{\frac{Et^2}{12\ \rho\left(1-\nu^2\right)}}$$

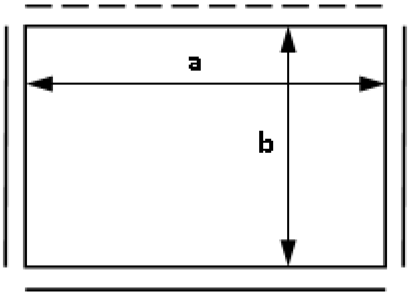
E=Young's modulus, t=thickness, ρ=density, a=length, b=width, μ=a/b, ν=Poisson's ratio (see below for values of λ)



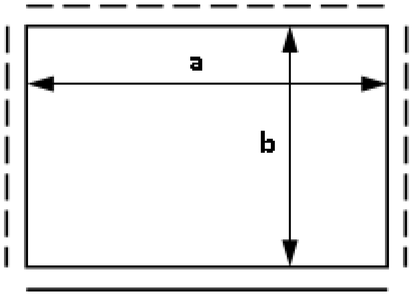
$$\lambda = 1.57\left(1+\mu^2\right)$$



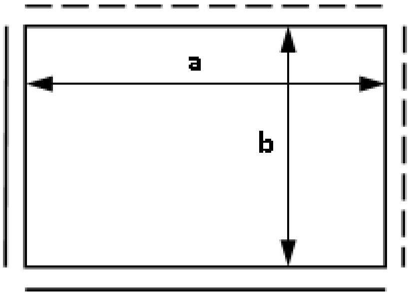
$$\lambda = 1.57\sqrt{\left(1+2.5\ \mu^2+5.14\ \mu^4\right)}$$



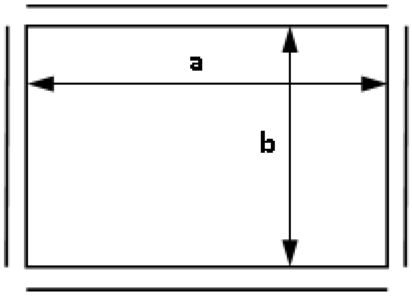
$$\lambda = 1.57\sqrt{\left(5.14+2.92\ \mu^2+2.44\ \mu^4\right)}$$



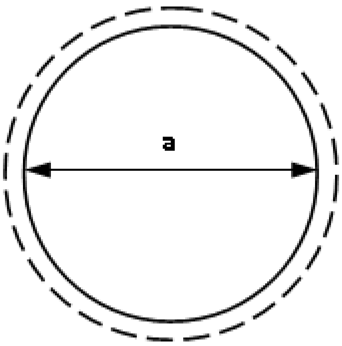
$$\lambda = 1.57\sqrt{\left(1+2.33\ \mu^2+2.44\ \mu^4\right)}$$



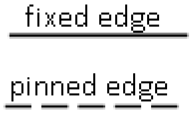
$$\lambda = 1.57\sqrt{\left(2.44+2.72\ \mu^2+2.44\ \mu^4\right)}$$

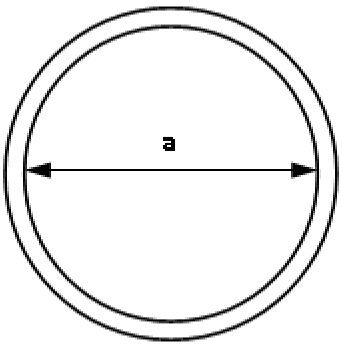


$$\lambda = 1.57\sqrt{\left(5.14+3.13\ \mu^2+5.14\ \mu^4\right)}$$



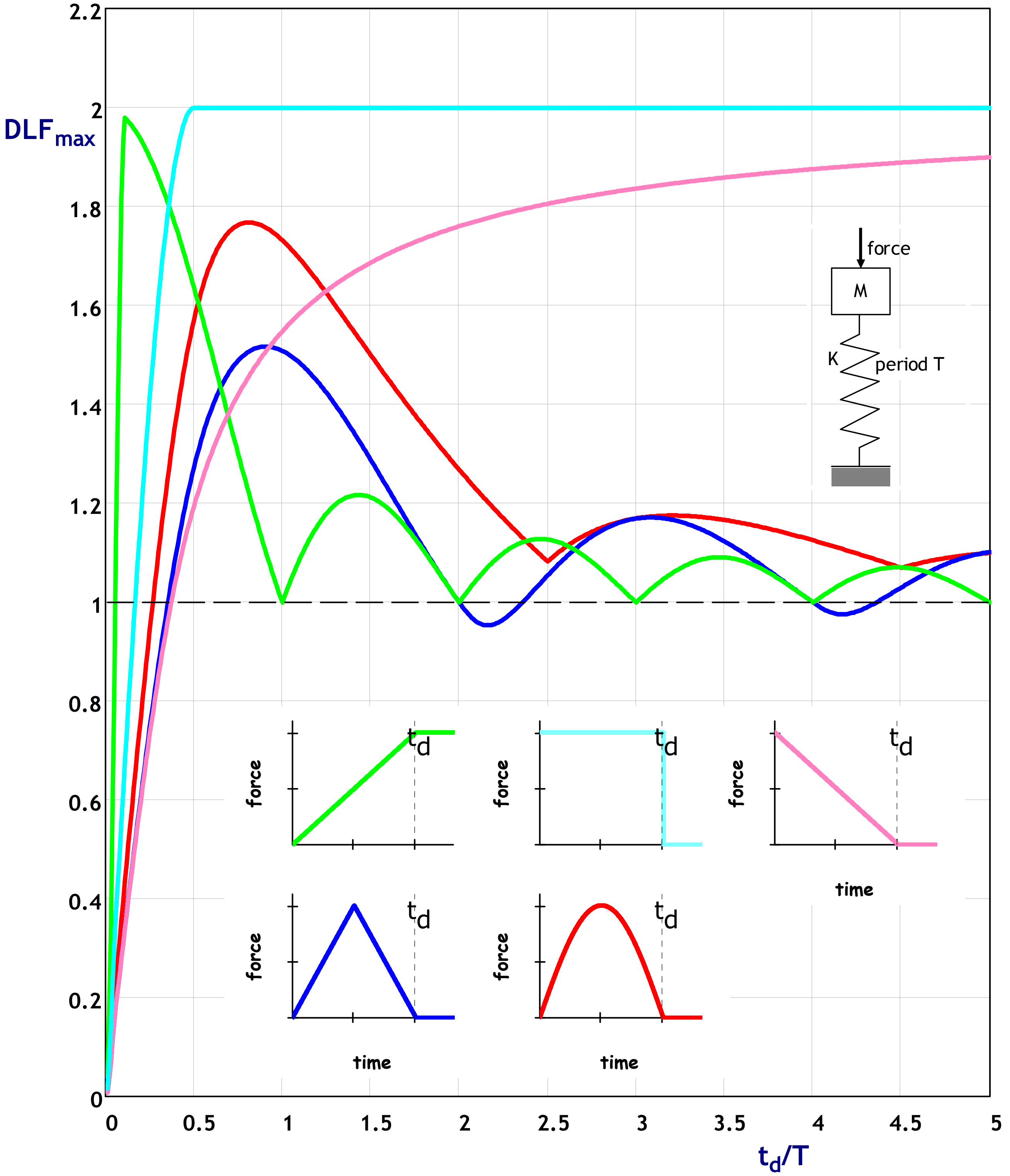
$$\lambda = 3.25$$





$$\lambda = 6.53$$

Maximum Response of an Undamped SDOF Elastic System Subject to Various Load Pulses



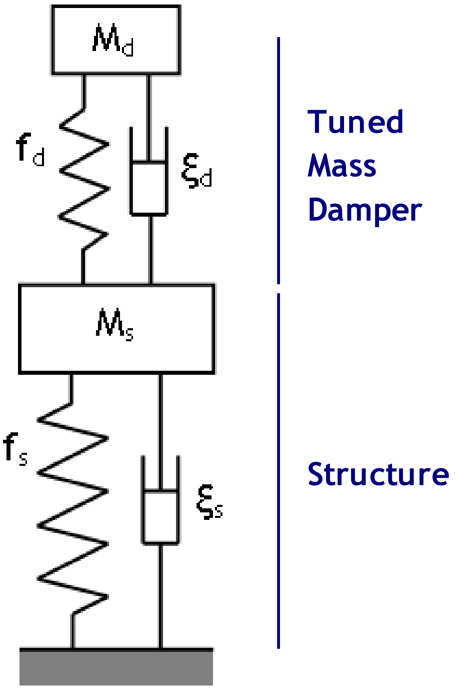
In the derivation of these charts no damping has been included because it has no significant effect. The maximum DLF usually corresponds to the first peak of response, and the amount of damping normally encountered in structures is not sufficient to sigificantly decrease this value.

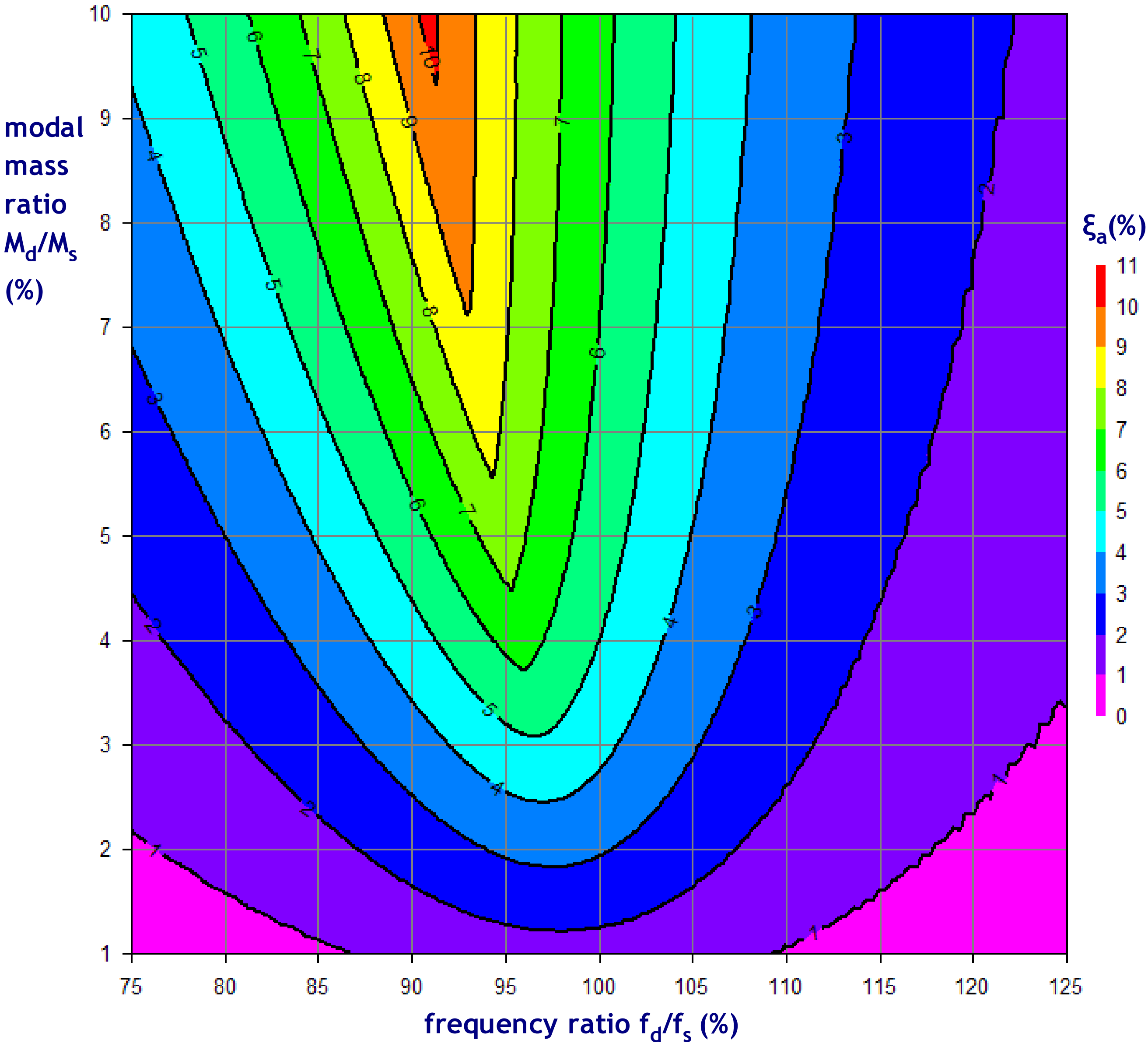
Tuned Mass Dampers

The TMD design chart below gives the additional overall damping ratio (ξ_a) provided by a TMD when attached to a structure with inherent damping ξ_s.

The chart is based on a TMD with ξ_d=15% and a structure with ξ_s in the range 0.5% - 2.5%. It also assumes that the TMD is located at the position of maximum response of the mode being damped.

The overall damping becomes ξ_s + ξ_a.





Tuned Mass Dampers

If the TMD can be tuned then the TMD chart shows that the additional damping is roughly

$$\xi_a = \frac{M_d}{M_s}$$

Maximum deflection of TMD relative to structure

$$xrel_d = \frac{1}{2\xi_d} = \frac{\pi}{\delta_d}$$

Optimum TMD frequency

$$f_d = \frac{f_s}{1+\mu} \quad \text{with} \quad \mu = \frac{M_d}{M_s}$$

Optimum TMD damping ratio

$$\xi_d = \sqrt{\frac{3\ \mu}{8(1+\mu)^3}}$$

Wind-induced vortex shedding

Critical wind speed for vortex shedding

$$v_{crit} = \frac{f_n\ D}{St}$$

f_n=natural frequency, D=across-wind dimension, St=Strouhal number.

For a circular cylinder St is approximately 0.2 and therefore $v_{crit} = 5\ f_n\ D$

The susceptibility of vortex-induced vibrations depends on the structural damping and the ratio of the structural mass to the fluid mass. This is expressed by the Scruton number (Sc), also known as the 'mass-damping parameter'.

$$Sc = \frac{2\ m_e\ \delta_s}{\rho_{air}\ D^2} \quad \text{with} \quad m_e = \frac{\int_0^L m(x)\ \phi(x)^2\ dx}{\int_0^L \phi(x)^2\ dx}$$

(equivalent mass per unit length)

Wind-induced galloping

Critical wind speed for galloping

$$v_{crit} = \frac{2\ Sc\ f_n\ D}{\frac{dC_y}{d\alpha}}$$

$dC_y/d\alpha$ is the rate of change of the lateral force coefficient with angle of attack

A section <u>is susceptible</u> to galloping if $\frac{dC_y}{d\alpha} > 0$ note that

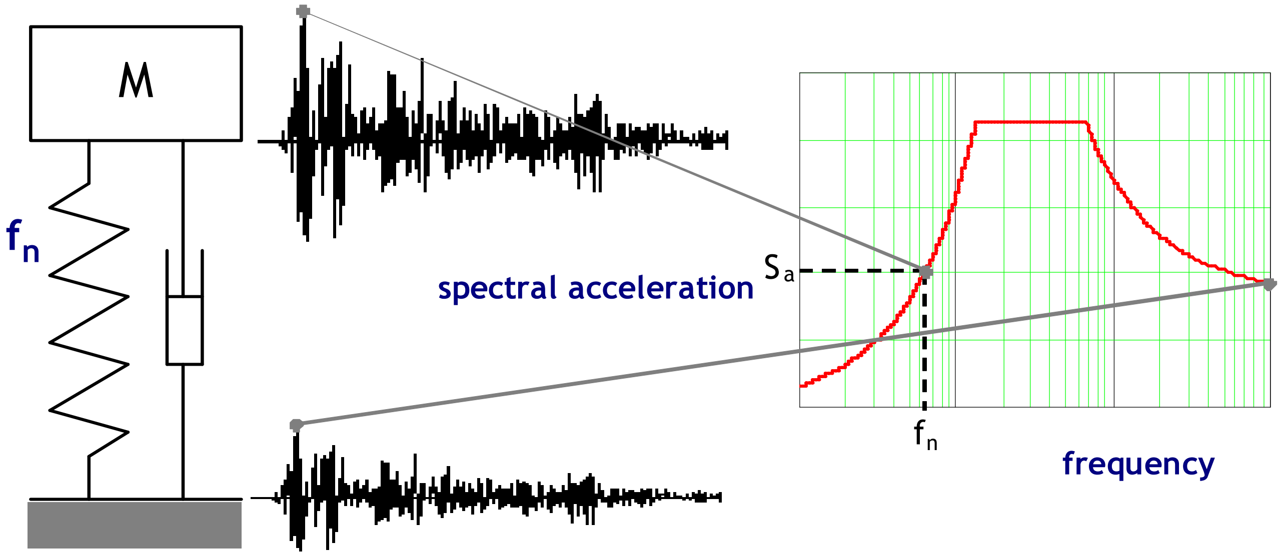
$$\frac{dC_y}{d\alpha} = -\left(\frac{dC_L}{d\alpha} + C_D\right)$$

with C_L=lift coefficient, C_D=drag coefficient

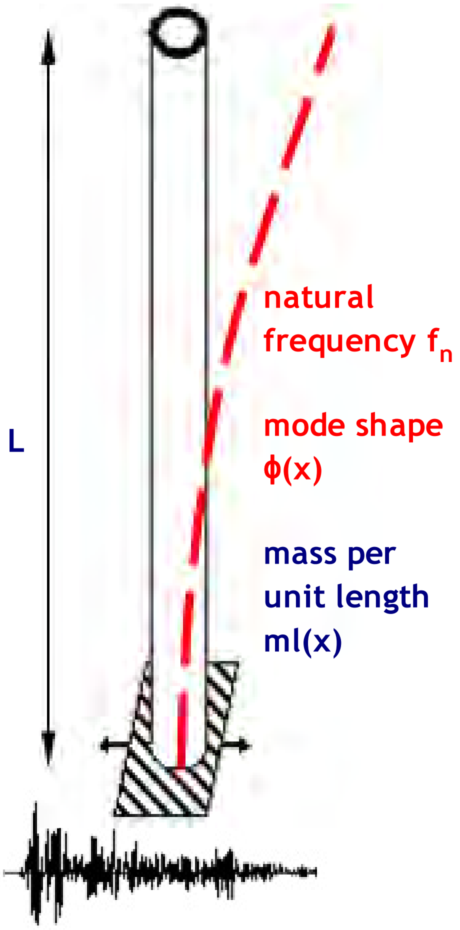
Seismic Response Spectrum Analysis

Relationship between SDOF response and structure response for each mode.

Single Degree of Freedom



Structure with distributed mass/stiffness



Generalized mass (modal mass)

$$M_G = \int_0^L \phi(x)^2 \cdot ml(x)\, dx$$

Participation factor

$$\Gamma = \frac{\int_0^L \phi(x) \cdot ml(x)\, dx}{\int_0^L \phi(x)^2 \cdot ml(x)\, dx}$$

Maximum acceleration at point i

$$a_i = S_a \cdot \Gamma \cdot \phi(x_i)$$

Effective mass

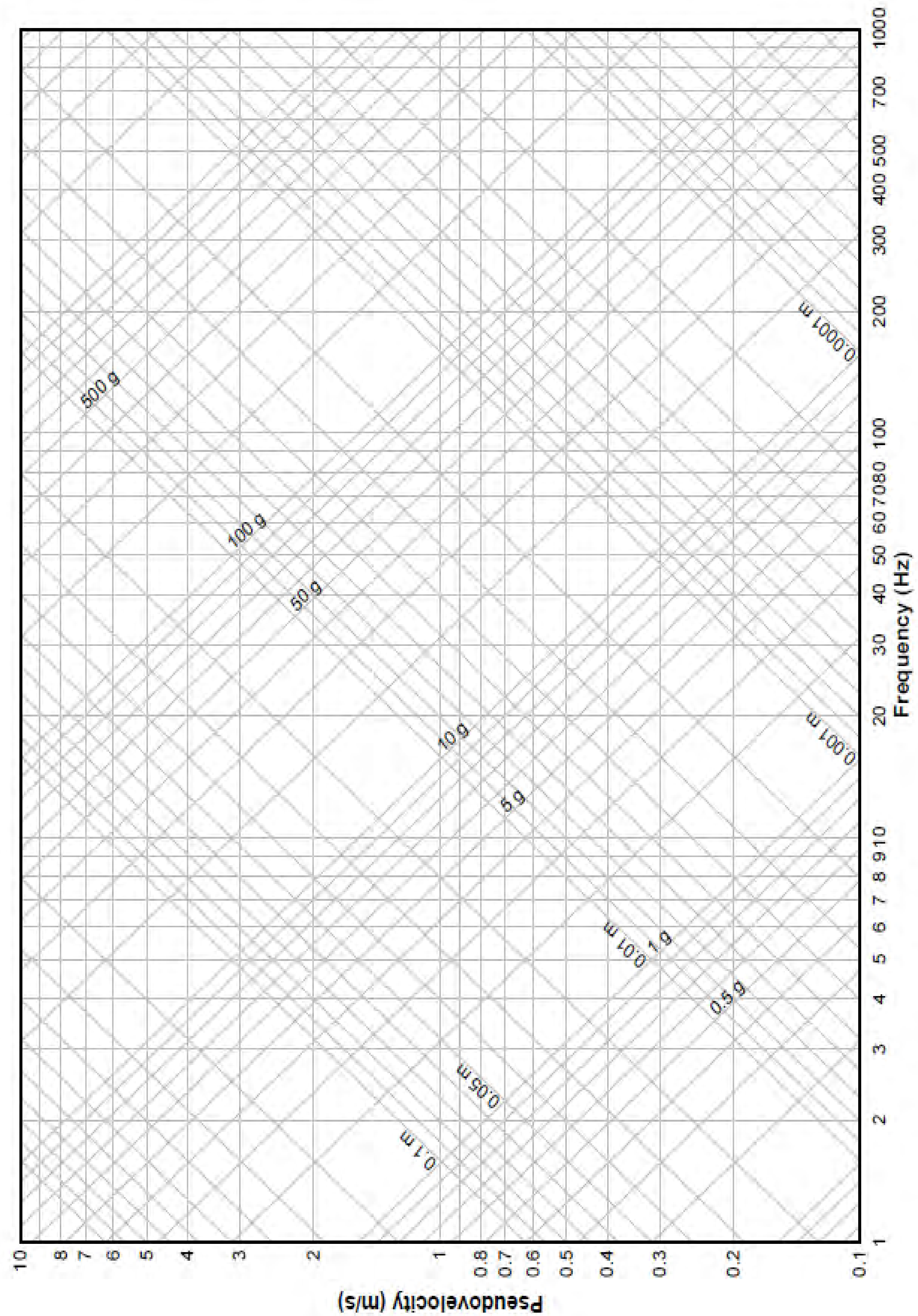
$$M_E = \frac{\left(\int_0^L \phi(x) \cdot ml(x)\, dx\right)^2}{\int_0^L \phi(x)^2 \cdot ml(x)\, dx}$$

Base shear

$$SF_{base} = S_a \cdot M_E$$

The maximum response from each of the modes can be combined using the 'Square Root Sum of Squares' (SRSS) or 'Complete Quadratic Combination' (CQC) method.

Tripartite Graph Paper



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Terminology

δ	logarithmic decrement
Δf	frequency interval
ξ	damping ratio
ω	circular frequency ($2\pi f$)
v	Poisson's ratio
Γ	participation factor
ϕ	mode shape
μ	aspect ratio, mass ratio
ρ	density
Δ	self-weight deflection
a	acceleration
C	damping coefficient
C_{crit}	critical damping coefficient
C_D	drag coefficient
C_L	lift coefficient
C_y	lateral force coefficient
D	across-wind dimension
DMF	dynamic magnification factor
E	Young's modulus
f	frequency
f_d	damped natural frequency
f_i	natural frequency of mode i
f_n	undamped natural frequency
g	acceleration due to gravity
h	plate thickness
H	height
J	mass moment of inertia
K	stiffness
L	length
ml	mass per unit length
M	mass
M_E	effective mass
M_G	generalized mass
N	number of cycles
SDOF	single degree of freedom
Sc	Scruton number
St	Strouhal number
t	time, thickness
T	period, tension
x	displacement

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